MMP Learning Seminar.

Week 50.

Contents:

Finiteness of B-representations.

Finiteness of B-representations:

Definition: (X,Δ) dlt projective pair.

$\mathrm{Bir}(X,\Delta)$ the group of all birational maps g of X.

such that if we take a common resolution:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "g"] & & X \end{tikzcd}$$

$$\Gamma \sim m(K_X+\Delta)$$
$$p^*\Gamma \sim mp^*(K_X+\Delta) = mq^*(K_X+\Delta)$$
$$p^*\Gamma \sim mq^*(K_X+\Delta)$$
$$q_*p^*\Gamma \sim m(K_X+\Delta)$$

Then $p^*(K_X+\Delta) = q^*(K_X+\Delta)$. The induced homomorphism

$$\rho_m : \mathrm{Bir}(X,\Delta) \longrightarrow \mathrm{Aut}(H^0(X,\mathcal{O}_X(m(K_X+\Delta))))$$

is called the B-representation of (X,Δ)

Theorem 1.2: Given a projective dlt pair (X,Δ) such that $K_X+\Delta$ is semiample $\mathbb{Q}$-divisor. There exists m s.t the image of the B-representations:

$$\rho_M : \mathrm{Bir}(X,\Delta) \longrightarrow \mathrm{Aut}(H^0(X,\mathcal{O}_X(M(K_X+\Delta))))$$

is finite for every M divisible by m.

Finite of B-representations:

$|m(K_X+\Delta)|$ is base point free and it induces an algebraic fibration $f: X \longrightarrow Y$ so that.

$$Y = \mathrm{Proj} \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(dm(K_X+\Delta))).$$

We can write $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + B + J)$.

$B \geq 0$ is the boundary part it depends on sing of the fibers.

$$B = \sum_{\substack{P \subseteq Y \\ \text{prime}}}' (1 - \mathrm{lct}((X,\Delta); f^*P))\, P$$

over the generic point of P.

J is the moduli part. is a b-divisor which is nef.

its definition comes from VHS.

Action on Y:

For every $g \in \mathrm{Bir}(X, \Delta)$ and $d \geq 0$, we have a homomorphism $\rho_{dm} : \mathrm{Bir}(X, \Delta) \longrightarrow \mathrm{Aut}_{\mathbb{C}}\, H^0(X, \mathcal{O}_X(dm(K_X + \Delta)))$.

Hence, we have an induced homomorphism

$$\chi : \mathrm{Bir}(X, \Delta) \longrightarrow \mathrm{Aut}(Y).$$

For any $g \in \mathrm{Bir}(X, \Delta)$, there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\chi(g)"] & Y \end{tikzcd}$$

For any d, we have an exact sequence:

$$1 \longrightarrow G \longrightarrow \rho_{dm}(\mathrm{Bir}(X, \Delta)) \longrightarrow \chi(\mathrm{Bir}(X, \Delta)) \longrightarrow 1$$

$G \subseteq H^0(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$ is a finite subgroup.

Theorem 3.1: The image $\chi(\mathrm{Bir}(X, \Delta)) \subseteq \mathrm{Aut}(Y)$ is finite.

Lemma 3.2: The image of $\chi(\mathrm{Bir}(X, \Delta))$ is contained in $\mathrm{Aut}(Y, B)$.

Heuristic of the proof:

$$\begin{array}{c} X \\ \Big\downarrow f \\ Y \end{array} \qquad K_X + \Delta \sim_{\mathbb{Q}} f^*(\underbrace{K_Y + B + J}_{\text{ample.}})$$

An element $g \in \mathrm{Bir}(X, \Delta)$ induces an element

$\chi(g) \in \mathrm{Aut}(Y, B)$., let's say that $\chi(g) \in \mathrm{Aut}(Y, B+J)$

$K_Y + B + J$ is ample "it behaves like a variety of gen type".

Philosophy: The subgroup $\mathrm{Aut}(Y, B+J)$ of $\mathrm{Aut}(Y, B)$ which "respect" J should be finite

Hodge theoretic construction:

(X, Δ) lc $\quad f : X \longrightarrow Y$ fibration of relative dim n.

$K_X + \Delta \sim_{\mathbb{Q}, Y} 0$.

$p : W \longrightarrow X$, $\quad p^*(K_X + \Delta) = K_W + E + F - G$, $\{F\} = F$

$aF \in \mathbb{Z}$, $\phi : W \longrightarrow Y$, ϕ is smooth over Y°

We may assume $(K_X + \Delta)|_{X^\circ} \sim_{\mathbb{Q}} 0$ by shrinking Y°

$V^\circ = \omega_{W^\circ}^{-1}(G^\circ - E^\circ)$ so that $V^{\circ \otimes a} \simeq \mathcal{O}_{W^\circ}(aF^\circ)$

$\hookrightarrow$ This data defines a local system $\mathbb{V}^\circ$ on $W^\circ \setminus \operatorname{Supp}(E^\circ \cup F^\circ)$

μ_a - cover $\pi : W' \longrightarrow W^\circ$, $\quad E' = \operatorname{red} \pi^* E^\circ$

$$\pi_*(\mathbb{C}|_{W' \setminus E'}) = \bigoplus_i \pi_*(\mathbb{C}|_{W' \setminus E'})^{(i)}$$

$\mathbb{V}^\circ$ isomorphic to the restriction of $\pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$,

we denote $\mathbb{V} = \pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$.

$(R^n \phi_* \mathbb{V})^{\otimes \alpha}$ is a local system on Y°.

$\phi' : W' \longrightarrow Y^\circ$, then $R^n \phi'_* \mathbb{V}$ is a direct summand of $R^n \phi'_* \mathbb{C}|_{W' \setminus E'}$ which carries a variation of mixed Hodge structure. The bottom piece of the Hodge filtration:

$$\begin{array}{ccc} F^n R^n \phi'_* (\mathbb{C}|_{W' \setminus E'}) & \simeq & \phi'_* \omega_{W'/Y}(E') \\ \cup\! | & & \cup\! | \\ F^n R^n \phi_* (\mathbb{V}|_{W^\circ \setminus E^\circ}) & \simeq & \phi_* \mathcal{O}_{W^\circ}(G^\circ) \end{array}$$

$\leftarrow$ line bundle on Y°

$L \subseteq W_{n+i}(R^n \phi_* \mathbb{V})$ but $L \nsubseteq W_{n+i-1}(R^n \phi_* \mathbb{V})$.

$\mathbb{H}$. smallest pure sub-$\mathbb{Q}$-VHS of $Gr^W_{n+i}(R^n \phi_* \mathbb{V})$ which contains L.

Lemma: $\mathbb{H}$ does not depend on the choice of the resolution

Finiteness of B-representations:

Proposition 3.3:

(1) Let $g \in Bir(X, \Delta)$. If we assume $\mathbb{H}$ is defined over an open set $Y^\circ \subseteq Y$. Then over $Y^\circ \cap \chi(g)^{-1}(Y^\circ)$, there is an isomorphism $i_g : \chi(g)^* \mathbb{H} \cong \mathbb{H}$.

(2) Let $g_1, g_2 \in Bir(X, \Delta)$, then over

$$Y^\circ \cap \chi(g_1)^{-1}(Y^\circ) \cap \chi(g_2)^{-1}(Y_0) \quad \text{we have}$$

$$i_{g_1} \circ i_{g_2} = i_{g_1 \circ g_2}.$$

Proof: Let W be a common resolution:

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, dashed, "g"] \arrow[d, "f"'] & & X \arrow[d, "f"] \\ Y \arrow[rr, "\chi(g)"] & & Y \end{tikzcd}$$

$$p_1^*(K_X + \Delta) = p_2^*(K_X + \Delta),$$

we can shrink Y°, so that $R^n \phi_{1*}(\mathbb{V})$ and $R^n \phi_{2*}(\mathbb{V})$ are defined over $Y^\circ \cap \chi(g)^{-1}(Y^\circ)$.

$\chi(g)^* L \cong L$.

Then, the isomorphism $\chi(g)^* : R^n \phi_{1*}(\mathbb{V}) \longrightarrow R^n \phi_{2*}(\mathbb{V})$ sends L to L, so it must send $\mathbb{H}$ to $\mathbb{H}$. $\square$.

Remark: $g \in \mathrm{Bir}(X, \Delta)$

$\chi(g)^* L \cong L.$

$\chi(\mathrm{Bir}(X,\Delta))$ is a subgroup of $PGL(N)$

$G := \overline{\chi(\mathrm{Bir}(X,\Delta))}$ algebraic closure of $\chi(\mathrm{Bir}(X,\Delta))$ in $PGL(N)$.

by contradiction, assume G inf.

G is a linear alg group, so it either contains $\mathbb{G}_a$ or $\mathbb{G}_m$.

Claim: G contains no sub-torus.

Torus action Lemma:

Lemma: (Y,B) sub-lc, $\psi: \mathbb{G}_m \times (Y,B) \longrightarrow (Y,B)$ faithful torus action. For $t \in Y$ general, if we denote $\psi_t: \mathbb{P}^1 \times \{t\} \longrightarrow Y$, the closure of the orbit, then

$$\deg \psi_t^* (K_Y + B) \leq 0.$$

Proof: $Y' \xrightarrow{\pi} Y$ log resol of (Y,B).

$$\mathbb{G}_m \cdot t \cap (\mathrm{supp}(B) \cup Ex(\pi)) = \emptyset.$$

$$\mathbb{G}_m \longrightarrow Y \setminus \mathrm{supp}(B) \cup \pi(Ex(\pi)).$$

$$\phi_t: \mathbb{P}^1 \longrightarrow Y'.$$

$\phi_T: \mathbb{P}^1 \times T \longrightarrow Y'$ generically finite

- $\phi_T|_{t \times \mathbb{P}^1} = \phi_t$
- $T \subseteq Y'$ open.

On the other hand, $\pi^*(K_Y + B) \leq K_{Y'} + B'$

We conclude that:

$$\phi_T^*(K_{Y'} + B') \leq K_{T\times\mathbb{P}^1} + T\times\{0\} + T\times\{\infty\}$$

Therefore, we conclude that:

$$\deg \psi_t^*(K_Y + B) \leq \deg \phi_t^*(K_{Y'} + B') \leq$$

$$\mathbb{P}^1 \cdot (K_{T\times\mathbb{P}^1} + T\times\{0\} + T\times\{\infty\}) = 0.$$

$\square$.

Theorem: G does not contain $\mathbb{G}_m$.

Proof: $\tilde{Y} = \bigcup_{g \in \mathcal{X}(Bir(X,\Delta))} g(Y_0)$.

$\mathbb{H}$ is non-degenerate over $\tilde{Y}$.

$o: \mathbb{P}^1 \longrightarrow Y$ general orbit, $o^* \mathbb{H}$ well-defined over $\mathbb{G}_m$

$Y' \xrightarrow{\pi} Y$ G-equivariant resolution of $(Y, Y \setminus \tilde{Y})$

$\mathbb{H}' = \pi^* \mathbb{H}$., defined outside the snc locus.

$\pi^*(K_Y + B + J) = K_{Y'} + B' + J'$, J' nef, (Y', B') sub-lc

$\phi: Z \longrightarrow Y'$ branched cover $\phi^* \mathbb{H}'$ has unipotent monodromy and extends to Z. $\phi_C: C \longrightarrow \mathbb{P}^1$ normalization of $\mathbb{P}^1 \times_Y Z$. of degree d.

Set $J_Z = \overline{E}^{n+1,0}(\phi^* \mathbb{H}')$, then

$$J_C = \overline{E}^{n+1,0}(\phi^* \mathbb{H}'|_C) = J_Z|_C.$$

$o_j^*(\mathbb{H})$ is a non-degenerate HS on $\mathbb{G}_m$ + unipotent monodromy at $\{0\}$ and $\{\infty\}$.

By Deligne's semisimple Theorem, $o_j^* \mathbb{H}$ is trivial.

$E^{n+1,0}(o_j^*(\mathbb{H})) = \mathcal{O}_{\mathbb{P}^1}(J_d)$, we have:

$$\frac{1}{d} d_* J_d \sim_{\mathbb{Q}} \frac{1}{\deg \phi_c} \phi_{c*} J_c \sim_{\mathbb{Q}} J'|_{\mathbb{P}^1} = 0$$

We conclude:

$$\deg (o^*(K_Y + B + J)) = \deg (o^*(K_Y + B))$$

$$\leq \deg (K_{Y'} + B') \leq 0.$$

$\rightarrow\leftarrow$

$\square$